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On the Role and Compensation of Distance Mismatches in Rigid Formation Control for Second-Order Agents ⋆

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Abstract:
This paper presents a robustness analysis of gradient-based formation control law for second-order agents subjected to distance mismatches or constant disturbances. It is shown that, akin to the first-order case, the existence of these mismatches introduces two undesired group behaviors: a distorted final shape and a stationary group motion. We show that such undesired properties can be compensated by combining the gradient-based rigid formation control law and our proposed distributed estimators.

Keywords: Formation Control, Rigid Formation, Motion Control, Second-Order dynamics.

1. INTRODUCTION

Maintaining a robotic formation has been one of important features in the operational of cooperative robots. This is highly relevant, for instance, during the exploration and surveillance of terrain (Cesare et al. (2015), Burgard et al. (2000)), for achieving energy-efficiency in group motion Tsugawa et al. (2011), for carrying heavy loads by team of robots Palunko et al. (2012), and many other group tasks. In these applications, gradient-based formation control law has been widely used due to its simplicity and ease of implementation.

One of the common assumptions in the derivation of these gradient-based control laws are that each robotic agent is modeled by a single integrator. It implies that the control action takes place in the velocity space, or in other words, it is assumed that we can instantaneously control the velocity of the robots. This assumption seems mild but it is not applicable to a wide-range of Euler-Lagrange systems where the control takes place in the acceleration space. However, by considering such an assumption, a simple gradient-based formation control using local distance information and local coordinates has been proposed and rigorously analyzed in literature. We refer interested readers to the works in Bai et al. (2011), Olfati-Saber and Murray (2002), Krick et al. (2009), Yu et al. (2009), Cao et al. (2011). It has been shown that such formation control law can guarantee the exponential stability of the desired shape (Sun et al. (2015), Sun and Anderson (2015)).

Despite the exponential stability property of the desired shape, it is not robust against constant disturbances in the proximity sensors or mismatches in the desired distance between communicating first-order agents, as shown recently in Mou et al. (2015) and Helmke et al. (2014). This constant bias introduces two undesired group behaviours, namely, distorted final formation shape and stationary group motion. This open problem was first tackled in Garcia de Marina et al. (2015a) using distributed estimators that can fully compensate the unknown constant disturbances or distance mismatches so that the group converges exponentially to the desired shape, without any distortion or undesired collective motion. Interestingly, when one looks from a different perspective where new control variables replace the distance mismatches, we can solve collective motion of a rigid formation problem with rotational and translational group motion as proposed very recently in Garcia de Marina et al. (2015b); a feat that cannot be done via the standard leader-follower approach with the use of estimators in all followers as pursued in Bai et al. (2011). The same approach can be used to solve the tracking and enclosing of a free target by a group of robots that is *not* necessarily forming a circle, as commonly considered in literature.

In this paper, we extend the aforementioned works in (Mou et al. (2015), Helmke et al. (2014), Garcia de Marina et al. (2015a)) from the first-order agent case to the second-order one. As mentioned before, this is more applicable since many robotic systems are described by Euler-Lagrange equations which correspond to second-order agents. In this case, the resulting formation control law can directly be used as the *desired acceleration* in a guidance system feeding the tracking controller of a mechanical system as the one proposed for quadrotors in Mellinger et al. (2012) or for marine vessels in Fossen (2002). The robustness stability analysis of the closed-loop system for second-order agents, as discussed in this paper, cannot follow the same steps as those used in (Mou et al. (2015), Helmke et al. (2014)). In particular, the error system that is considered in these papers for stability analysis

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is an autonomous system, which is not the case for the second-order agent as shown later in Section 3. Hence, we need to establish further additional steps in deriving the robustness results.

In the first part of the paper, we study the robustness of the gradient-based formation control for second-order agents with respect to constant distance mismatches. In the second part, we propose a distributed estimator design that can eliminate the undesired behaviors due to the existence of these mismatches. The rest of the paper is organized as follows. We review some standard definitions and notations in Section 2. In Section 3, we present our first main result where we provide a robustness analysis in the formation of second-order agents using gradient-based formation control under the presence of distance mismatches. In Section 4, we propose the design of distributed estimators in dealing with the distance mismatches. We provide simulation results in Section 5.

2. PRELIMINARIES

In this section, we introduce some notations and concepts related to graphs and rigid formations. For a given matrix $A \in \mathbb{R}^{n\times p}$, define $\overline{A} \triangleq A \otimes I_m \in \mathbb{R}^{nm\times pm}$, where the symbol $\otimes$ denotes the Kronecker product, $m = 2$ for $\mathbb{R}^2$ or otherwise 3 for $\mathbb{R}^3$, and I_m is the m-dimensional identity matrix. For a stacked vector $x \triangleq \begin{bmatrix} x_1^T & x_2^T & \dots & x_k^T \end{bmatrix}^T$ with $x_i \in \mathbb{R}^n, i \in \{1, \dots, k\}$, we define the diagonal matrix $D_x \triangleq \text{diag}\{x_i\}_{i\in\{1,\dots,k\}} \in \mathbb{R}^{kn\times k}$. We denote by $|\mathcal{X}|$ the cardinality of the set $\mathcal{X}$ and by $||x||$ the Euclidean norm of a vector x. We use $\mathbf{1}_{n\times m}$ and $\mathbf{0}_{n\times m}$ to denote the all-one and all-zero matrix in $\mathbb{R}^{n\times m}$ respectively and we will omit the subscript if the dimensions are clear from the context.

2.1 Graphs and Minimally Rigid Formations

We consider a formation of $n \geq 2$ autonomous agents whose positions are denoted by $p_i \in \mathbb{R}^m$. The agents can measure their relative positions with respect to its neighbors. This sensing topology is given by an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ with the vertex set $\mathcal{V} = \{1, \dots, n\}$ and the ordered edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$. The set $\mathcal{N}_i$ of the neighbors of agent i is defined by $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. We define the elements of the incidence matrix $B \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ for $\mathbb{G}$ by

$$b_{ik} \triangleq \begin{cases} +1 & \text{if} \quad i = \mathcal{E}_k^{\text{tail}} \\ -1 & \text{if} \quad i = \mathcal{E}_k^{\text{head}} \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{E}_k^{\text{tail}}$ and $\mathcal{E}_k^{\text{head}}$ denote the tail and head nodes, respectively, of the edge $\mathcal{E}_k$, i.e. $\mathcal{E}_k = (\mathcal{E}_k^{\text{tail}}, \mathcal{E}_k^{\text{head}})$. A *framework* is defined by the pair $(\mathbb{G}, p)$, where $p = \text{col}\{p_1, \dots, p_n\}$ is the stacked vector of the agents' positions $p_i, i \in \{1, \dots, n\}$. Then we embed the positions of the agents in the graph's nodes and the available relative measurements in the graph's edges. With this at hand, we define the stacked vector of the measured relative positions by

$$z = \overline{B}^T p,$$

where each vector $z_k = p_i - p_j$ in z corresponds to the relative position associated with the edge $\mathcal{E}_k = (i, j)$.

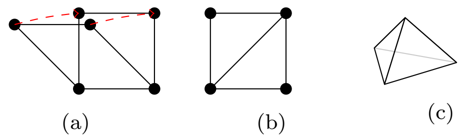

Fig. 1. a) The square without an inner diagonal is not rigid since we can smoothly move the top two nodes while keeping the other two fixed without breaking the distance constraints; b) The square can be done locally minimally rigid in $\mathbb{R}^2$ if we add an inner diagonal; c) The tetrahedron in $\mathbb{R}^3$ is globally minimally rigid.

For a given stacked vector of desired relative positions $z^* = [\, z_1^{*T} \; z_2^{*T} \; \dots \; z_{|\mathcal{E}|}^{*\,T} \,]^T$, the resulting set $\mathcal{Z}$ of the possible formations with the same shape is defined by

$$\mathcal{Z} \triangleq \left\{ \left(I_{|\mathcal{E}|} \otimes \mathcal{R} \right) z^* \right\}, \tag{1}$$

where $\mathcal{R}$ is the set of all rotational matrices in 2D or 3D. Roughly speaking, $\mathcal{Z}$ consists of all formation positions that are obtained by rotating z^*.

Let us now briefly recall the notions of infinitesimally rigid framework and minimally rigid framework from Anderson et al. (2008). Define the edge function $f_{\mathbb{G}}$ by $f_{\mathbb{G}}(p) = \underset{k}{\text{col}} \left(\|z_k\|^2 \right)$ where the operator col defines stacked column vector and we denote its Jacobian by $R(z)$ and is called the *rigidity matrix*. A framework $(\mathbb{G}, p)$ is *infinitesimally rigid* if $\text{rank}R(z) = 2n - 3$ when it is embedded in $\mathbb{R}^2$ or if $\text{rank}R(z) = 3n - 6$ when it is embedded in $\mathbb{R}^3$. Additionally, if $|\mathcal{E}| = 2n-3$ in the 2D case or $|\mathcal{E}| = 3n-6$ in the 3D case then the framework is called *minimally rigid*.

Roughly speaking, the only motions that we can perform over the agents in a minimally rigid framework, while they are already in the desired shape, are the ones defining translations and rotations of the whole shape. Some graphical examples of minimally rigid frameworks are shown in Figure 1. If $(\mathbb{G}, p)$ is minimally rigid, then, similar to the above, we can define the set of resulting formations $\mathcal{D}$ by

$$\mathcal{D} \triangleq \left\{ \; z \; \middle| \; ||z_k|| = d_k, k \in \{1, \dots, |\mathcal{E}|\} \right\},$$

where $d_k = ||z_k^*||, k \in \{1, \dots, |\mathcal{E}|\}$.

Note that in general it holds that $\mathcal{Z} \subseteq \mathcal{D}$. For a desired shape, one can always design $\mathbb{G}$ to make the formation minimally rigid. In fact in $\mathbb{R}^2$, a minimally rigid framework with two or more vertices can always be constructed through the Henneberg construction Henneberg (1911). In $\mathbb{R}^3$ one can construct a set of minimally rigid frameworks via insertion starting from a tetrahedron, if each new added vertex with three new links forms another tetrahedron as well.

2.2 Frames of coordinates

It will be useful for describing the motions of the minimally rigid formation to define a frame of coordinates fixed to the formation itself. We denote by O_g the *global frame* of coordinates fixed at the origin of $\mathbb{R}^m$ with some arbitrary fixed orientation. In a similar way, we denote by O_b the *body frame* fixed at the centroid p_c of the desired rigid formation. Furthermore, if we rotate the rigid formation with respect to O_g, then O_b is also rotated in the same manner. Let ${}^b p_j$ denote the position of agent j with respect to O_b. To simplify notation whenever we represent an agents' variable with respect to O_g, the superscript is omitted, e.g. $p_j \triangleq {}^g p_j$.

3. ROBUSTNESS ISSUES DUE TO MISMATCHES IN FORMATION GRADIENT-BASED CONTROL

3.1 Gradient Control

Consider a formation of n agents with the sensing topology $\mathbb{G}$ for measuring the relative positions among the agents. The agents are modelled by a second-order system given by

$$\begin{cases} \dot{p} = v \\ \dot{v} = u \end{cases}, \tag{2}$$

where u and v are the stacked vector of control inputs $u_i \in \mathbb{R}^m$ and vector of agents' velocity $v_i \in \mathbb{R}^m$ for $i = \{1, \ldots, n\}$ respectively.

In order to control the shape, for each edge $\mathcal{E}_k = (i, j)$ in the minimally rigid framework we assign the following potential function V_k

$$V_k(||z_k||) = \frac{1}{4}(||z_k||^2 - d_k^2)^2,$$

with the gradient along p_i or p_j given by

$$\nabla_{p_i} V_k = -\nabla_{p_j} V_k = z_k(||z_k||^2 - d_k^2).$$

In order to control the agents' velocities, for each agent i in the minimally rigid framework we assign the following potential function S_i

$$S_i(v_i) = \frac{1}{2}||v_i||^2,$$

with the gradient along v_i be given by

$$\nabla_{v_i} S_i = v_i.$$

One can check that for the potential function

$$\phi(p, v) = \sum_{i=1}^{|\mathcal{V}|} S_i + \sum_{k=1}^{|\mathcal{E}|} V_k, \tag{3}$$

the closed-loop system (2) with the control input

$$u = -\nabla_v \phi - \nabla_p \phi, \tag{4}$$

becomes the following dissipative Hamiltonian system Schaft (2006)

$$\begin{cases} \dot{p} = \nabla_v \phi \\ \dot{v} = -\nabla_v \phi - \nabla_p \phi. \end{cases} \tag{5}$$

Considering (3) as the storage energy function of the Hamiltonian system (5), one can show the local asymptotic convergence of the formation to the shape given by $\mathcal{D}$ and all the agents' velocities to zero Bai et al. (2011); Oh and Ahn (2014).

Let the following one-parameter family of dynamical systems $\mathcal{H}_\lambda$ given by

$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = -\begin{bmatrix} \lambda I_{m|\mathcal{V}|} & -(1-\lambda) I_{m|\mathcal{V}|} \\ (1-\lambda) I_{m|\mathcal{V}|} & I_{m|\mathcal{V}|} \end{bmatrix} \begin{bmatrix} \nabla_p \phi \\ \nabla_v \phi \end{bmatrix}, \tag{6}$$

where $\lambda \in [0, 1]$, which defines all convex combinations of the Hamiltonian system (5) and a gradient system. The family $\mathcal{H}_\lambda$ has two important properties summarized in the following lemma.

Lemma 1. Oh and Ahn (2014)

- For all $\lambda \in [0, 1]$, the equilibrium set of $\mathcal{H}_\lambda$ is given by the set of the critical points of the potential function ϕ, i.e. $E_{p,v} = \left\{ \begin{bmatrix} p^T & v^T \end{bmatrix}^T : \nabla \phi = 0 \right\}$.
- For any equilibrium $\begin{bmatrix} p^T & v^T \end{bmatrix}^T \in E_{p,v}$ and for all $\lambda \in [0, 1]$, the numbers of the stable, neutral, and unstable eigenvalues of the Jacobian of $\mathcal{H}_\lambda$ are the same and independent of λ.

This result has been exploited in Sun and Anderson (2015) in order to show the local exponential convergence of $z(t)$ and $v(t)$ to $\mathcal{D}$ and $\mathbf{0}_{|\mathcal{V}|\times 1}$ respectively. In the following brief exposition we revisit the result to show such exponential stability via a combination of Lyapunov argument and Lemma 1, which will play an important role in Section 3.2.

Define the distance error corresponding to the edge $\mathcal{E}_k$ by

$$e_k = ||z_k||^2 - d_k^2,$$

whose time derivative is given by $\dot{e}_k = 2 z_k^T \dot{z}_k$. Consider the following autonomous system derived from (6) with $\lambda = 0.5$

$$\dot{p} = -\frac{1}{2}\overline{B} D_z e + \frac{1}{2} v$$

$$\dot{z} = -\frac{1}{2}\overline{B}^T \overline{B} D_z e + \frac{1}{2}\overline{B}^T v \tag{7}$$

$$\dot{e} = -D_z^T \overline{B}^T \overline{B} D_z e + D_z^T \overline{B}^T v \tag{8}$$

$$\dot{v} = -\frac{1}{2}\overline{B} D_z e - \frac{1}{2} v, \tag{9}$$

where e is the stacked vector of e_k's for all $k \in \{1, \ldots, |\mathcal{E}|\}$. Define the speed of the agent i by

$$s_i = ||v_i||,$$

whose time derivative is given by $\dot{s}_i = \frac{v_i^T \dot{v}_i}{s_i}$. Their compact form involving all the agents can be written as

$$\dot{s} = D_{\tilde{s}} D_v^T \dot{v} = -\frac{1}{2} D_{\tilde{s}} D_v^T \overline{B} D_z e - \frac{1}{2} D_{\tilde{s}} D_v^T v, \tag{10}$$

where s and $\tilde{s}$ are the stacked vectors of s_i's and $\frac{1}{s_i}$'s for all $i \in \{1, \ldots, |\mathcal{V}|\}$ respectively. Now we are ready to show the local exponential convergence to the origin of the speed of the agents and the error distances in the edges.

Lemma 2. The origins $e = 0$ and $s = 0$ of the error and speed systems derived from (5) are locally exponentially stable if the given desired shape $\mathcal{D}$ is minimally rigid.

Proof. Consider e_λ and s_λ as the stacked vectors of the error signals e_k and speeds s_k derived from (6) for any $\lambda \in [0, 1]$, which includes the system (5) for $\lambda = 1$. Looking at the definition of e_k, we know that all the e_λ share the same stability properties by invoking Lemma 1, so s_λ as well.

Consider the following candidate Lyapunov function for the autonomous system (7)-(10) derived from (6) with $\lambda = 0.5$

$$V = \frac{1}{2}||e||^2 + ||s||^2,$$

whose time derivative satisfies

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= e^T\dot{e} + 2s^T\dot{s} \\
&= -e^T D_z^T \overline{B}^T \overline{B} D_z e + e^T D_z^T \overline{B}^T v - \underbrace{s^T D_{\tilde{s}} D_v^T}_{v^T} \overline{B} D_z e \\
&\quad - \underbrace{s^T D_{\tilde{s}}}_{\mathbf{1}_{1\times|\mathcal{V}|}} D_v^T v \\
&\leq -\sigma_{\min}||e||^2 - ||s||^2, \qquad (11)
\end{aligned}
$$

where $\sigma_{\min}$ is the minimum eigenvalue of $D_z^T \overline{B}^T \overline{B} D_z = R(z)R(z)^T$ in the compact set $\mathcal{Q} \triangleq \{e : ||e||^2 \leq \rho\}$ for some $\rho > 0$. Note that for minimally rigid frameworks $R(z)$ is full rank except for non-generic situations (e.g. collinear or coplanar alignments of the agents in $\mathbb{R}^2$ or $\mathbb{R}^3$). Therefore if the initial conditions for the error signal and the speed satisfy $||e(0)||^2 + ||s(0)||^2 \leq \rho$, then $\sigma_{\min} > 0$ for a sufficiently small ρ since $\mathcal{D}$ is minimally rigid. Hence we conclude the local exponential convergence of $e(t)$ and $s(t)$ to the origin.

Remark 3. It is worth noting that the region of attraction determined by ρ in the proof of Lemma 2 for $\lambda = 0.5$ might be different from the one for $\lambda = 1$, since Lemma 1 only refers to the Jacobian of (6), i.e. the linearization of the system about the equilibrium.

It can be concluded from the exponential convergence to zero of the speeds of the agents $s(t)$ that the formation will eventually stop. This implies that $p(t)$ will converge exponentially to a finite point in $\mathbb{R}^m$ as $z(t)$ converges exponentially to $\mathcal{D}$.

3.2 Robustness issues caused by mismatches

It is obvious somehow that for a general distance-based formation control problem with $n = 2$, if the two agents do not share the same prescribed distance to maintain, then an eventual steady-state motion will happen regardless of the dynamics of the agents since the agent with a smaller prescribed distance will chase the other one. Therefore, for $n > 2$ it would not be surprising to observe some collective motion in the steady-state of the formation if the neighboring agents do not share the same prescribed distance to maintain.

When two neighboring agents disagree on the desired squared distance d_k^2 in between, namely

$$
d_k^{2\,\text{tail}} = d_k^{2\,\text{head}} - \mu_k, \qquad (12)
$$

where $\mu_k \in \mathbb{R}$ is a constant mismatch, it can be checked that this disagreement leads to mismatched potential functions, therefore agents i and j do not share anymore the same V_k for $\mathcal{E}_k = (i, j)$, namely

$$
V_k^i = \frac{1}{4}(||z_k||^2 - d_k^2 + \mu_k)^2, \quad V_k^j = \frac{1}{4}(||z_k||^2 - d_k^2)^2,
$$

under which the control laws for agents i and j use the gradients of V_k^i and V_k^j respectively for the edge $\mathcal{E}_k = (i, j)$. In the presence of one mismatch in every edge, the control signal (4) can be rewritten as

$$
u = -v - \overline{B} D_z e - \overline{S}_1 D_z \mu, \qquad (13)
$$

where S_1 is constructed from the incidence matrix by setting its -1 elements to 0, and $\mu \in \mathbb{R}^{|\mathcal{E}|}$ is the stacked column vector of μ_k's for all $k \in \{1, \dots, |\mathcal{E}|\}$. Note that (13) can be also written as

$$
u = -v - \overline{B} D_z e - \overline{A}_1(\mu) z, \qquad (14)
$$

where the elements of A_1 are

$$
a_{ik} \triangleq \begin{cases} \mu_k & \text{if} \quad i = \mathcal{E}_k^{\text{tail}} \\ 0 & \text{otherwise.} \end{cases} \qquad (15)
$$

Inspired by Mou et al. (2015), we will show how μ can be seen as a parametric disturbance in an autonomous system whose origin is exponentially stable. Let the error signal e and the speed of the agents s derived from system (2) with the control input (4)

$$
\dot{e} = 2D_z^T \overline{B}^T v \qquad (16)
$$

$$
\dot{s} = -s - D_{\tilde{s}} D_v^T \overline{B} D_z e, \qquad (17)
$$

and define

$$
\alpha_{ki} = z_k^T v_i, \quad k \in \{1, \dots, |\mathcal{E}|\}, i \in \{1, \dots, |\mathcal{V}|\} \qquad (18)
$$

$$
\beta_{ij} = v_i^T v_j, \quad i, j \in \{1, \dots, |\mathcal{V}|\}, i \neq j. \qquad (19)
$$

We stack all the α_{ki}'s and β_{ij}'s in the column vectors $\alpha \in \mathbb{R}^{|\mathcal{E}||\mathcal{V}|}$ and $\beta \in \mathbb{R}^{\frac{|\mathcal{V}|(|\mathcal{V}|+1)}{2}}$ respectively and define $\gamma \triangleq \begin{bmatrix} e^T & s^T & \alpha^T & \beta^T \end{bmatrix}^T$. We know that for any minimally rigid framework, there exists a neighborhood $\mathcal{U}_z$ about this framework such that for all $z_k, z_l \in \mathcal{U}_z$ with $k, l \in \{1, \dots, |\mathcal{E}|\}$, we can write $z_k^T z_l$ by a smooth function $g_{kl}(e)$ (Mou et al. (2015)). Then using (16)-(19) we get

$$
\dot{\gamma} = f(\gamma), \qquad (20)
$$

which is an autonomous system whose origin is locally exponentially stable using the results from Lemmas 1 and 2. Obviously, in such a case, the following Jacobian evaluated at $\gamma = 0$

$$
J = \left.\frac{\partial f(\gamma)}{\partial \gamma}\right|_{\gamma=0},
$$

has all its eigenvalues in the left half complex plane. From the system (2) with control law (13) we can *extend* (20) but with a parametric disturbance μ because of the third term in (13), namely

$$
\dot{\gamma} = f(\gamma, \mu), \qquad (21)
$$

where $f(\gamma, \mathbf{0}_{|\mathcal{V}|\times 1})$ is the same as in (20) derived from the gradient controller. Therefore, for a sufficiently small $||\mu||$, the Jacobian $\left.\frac{\partial f(\gamma,\mu)}{\partial \gamma}\right|_{\gamma=0}$ is still a stable matrix since the eigenvalues of a matrix are continuous functions of its entries. Although the system (21) is still stable under the presence of a small disturbance μ, the equilibrium point is not the origin in general anymore but $\gamma(t) \to \hat{\gamma}(\mu)$ as t goes to infinity, where $\hat{\gamma}(\mu) \triangleq \gamma_\mu$ is a smooth function of μ with zero value if $\mu = \mathbf{0}_{|\mathcal{V}|\times 1}$ (Khalil and Grizzle (1996)). This implies that in general each component of e, s, α and β converges to a non-zero constant with the following two immediate consequences: the formation shape will be distorted, i.e. $e \neq 0$; and the agents will not remain stationary, i.e. $s \neq 0$. The meaning of having non-zero components in general in α and β is that the velocities of the agents have a fixed relation with the steady-state shape. If the disturbance $||\mu||$ is sufficiently small, then $||\hat{\gamma}(\mu)|| < \rho$ for some small $\rho \in \mathbb{R}^+$ implying that $||e(t)|| < \rho$, and if further ρ is sufficiently small, then the stationary distorted shape is also minimally rigid. In addition since the speeds of the agents converge to a constant (in general non-zero constant), then only translations and/or rotations of the stationary distorted shape can happen. We summarize in the following theorem.

Theorem 4. Consider system (2) with control input (13) where the desired shape for the formation is minimally rigid and μ is considered as a small parametric perturbation. Then, the formation will converge to a distorted minimally rigid shape, i.e. $e \neq 0$, and the agents will converge to a steady-state collective motion that can be captured by constants angular and translational velocities ${}^b\omega^*$ and ${}^bv_c^*$, respectively, for the distorted minimally rigid formation.

Proof. Since system (21), derived from (2) and (13), is self-contained and its origin is locally exponentially stable with $\mu = \mathbf{0}_{|\mathcal{E}|\times 1}$, then a small parametric perturbation μ in its Jacobian will not change the exponential stability property of (21) but its equilibrium point. Therefore $e(t) \to e_\mu$ as t goes to infinity, where $e_\mu \in \mathbb{R}^{|\mathcal{E}|}$ is non-zero. In addition if the norm of μ is sufficiently small, then the stationary distorted shape will still be minimally rigid. We also have that the elements of $s(t) \to s_\mu$ as t goes to infinity with $s_\mu \in \mathbb{R}^{|\mathcal{V}|}$ having all its elements non-negative and in general non-zero, implying that the agents will not stop in the steady-state. Since the steady-state shape of the formation locally converges to a minimally rigid one, from the error dynamics (16) we have that

$$D_{z(t)}^T \overline{B}^T v(t) = R\left(z(t)\right) v(t) \to \mathbf{0}_{m|\mathcal{V}|\times 1}, \quad t \to \infty,$$

therefore $v(t) \to v_\mu(t)$ as t goes to infinity, where the non-constant $v_\mu(t) \in \mathbb{R}^{|\mathcal{V}|}$ belongs to the null space of $R\left(z_\mu(t)\right)$, $z_\mu(t) \in \mathcal{Z}_\mu$ and the set $\mathcal{Z}_\mu$ is defined as in (1) but corresponding to the inter-distances of the distorted minimally rigid shape with $e = e_\mu$. Note that obviously, the evolution of $z(t)$ is a consequence of the evolution of agents' velocities in $v(t)$. The null space of $R\left(z_\mu\right)$ corresponds to the infinitesimal motions δp_i for all i while keeping the inter-distances in the distorted formation constant, namely

$$R(z_\mu)\delta p = R(z_\mu) v_\mu \, \delta t = \mathbf{0}_{m|\mathcal{V}|\times 1},$$

or in order words

$$v_i(t) \to v_{\mu_i}(t), \quad t \to \infty, \tag{22}$$

where the velocities $v_{\mu_i}(t)$'s for all the agents are the result of rotating and translating the steady-state distorted shape defined by $\mathcal{Z}_\mu$. This steady-state collective motion of the distorted formation can be represented by the rotational and translational velocities ${}^b\omega^*(t) \in \mathbb{R}^m$ and ${}^bv_c(t)^* \in \mathbb{R}^m$ (possibly not constant) at the centroid of the distorted rigid shape. Furthermore, by definition we have that $||v_{\mu_i}(t)|| = s_{\mu_i}$. Since the speed s_{μ_i} for agent i is constant but not its velocity $v_{\mu_i}(t)$, the acceleration $a_{\mu_i}(t) = \frac{\mathrm{d}v_{\mu_i}(t)}{\mathrm{d}t}$ is perpendicular to $v_{\mu_i}(t)$. The expression of $a_{\mu_i}(t)$ can be derived from (14) and is given by

$$a_{\mu_i}(t) = -v_{\mu_i}(t) - \sum_{k=1}^{|\mathcal{E}|} b_{ik} z_{\mu_k}(t) e_{\mu_k} + \sum_{k=1}^{|\mathcal{E}|} a_{ik} z_{\mu_k}(t), \tag{23}$$

where b_{ik} are the elements of the incidence matrix, and a_{ik} are the elements of the *perturbation* matrix A_1 as defined in (15). From (23) it is clear that the norm $||a_{\mu_i}(t)|| = \Gamma_i(\gamma_\mu)$ is constant. In addition since $a_{\mu_i}(t)$ is a continuous function, i.e. the acceleration vector cannot switch its direction, and it is perpendicular to $v_{\mu_i}(t)$, the only possibility for the distorted formation is to follow a motion described by constant velocities ${}^b\omega^*$ and ${}^bv_c^*$ at its centroid.

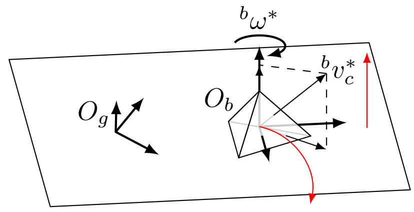

Fig. 2. The velocities ${}^b\omega^*$ and ${}^bv_c^*$ at the centroid of the tetrahedron rotates and translates the minimally rigid formation respectively. For having constant vectors ${}^b\omega^*$ and ${}^bv_c^*$ the formation describes a closed orbit in the plane where ${}^b\omega^*$ and ${}^bv_c^*$ are perpendicular (always the case in 2D formations) plus a constant drift along the projection of ${}^bv_c^*$ over ${}^b\omega^*$.

Remark 5. In particular, in 2D the distorted formation will follow a closed orbit if $\Gamma_i(\gamma_\mu) \neq 0$ for all i, or a constant drift if $\Gamma_i(\gamma_\mu) = 0$ for all i. This is due to the fact that in 2D, ${}^b\omega^*$ and ${}^bv^*$ are always perpendicular or equivalently $a_{\mu_i}(t)$ and $v_{\mu_i}(t)$ lie in the same plane. The resultant motion in 3D is the composition of a drift plus a closed orbit, since ${}^b\omega^*$ and ${}^bv_c^*$ are constant and they do not need to be perpendicular to each other as it can be noted in Figure 2.

Remark 6. Although the disturbance μ acts on the acceleration of second-order the agents, it turns out that the resultant collective motion has the same behavior as for having the disturbance μ acting in the velocity for first-order agents. A detailed description of such a motion related to the disturbance in first-order agents can be found in Sun et al. (2014); Garcia de Marina et al. (2015b).

4. ESTIMATOR-BASED GRADIENT CONTROL

Let us consider the following distributed control law with estimator

$$\begin{cases} \dot{\hat{\mu}} &= \hat{u} \\ u &= -v - \overline{B}D_z e - \overline{S}_1 D_z(\mu - \hat{\mu}) \end{cases},$$

where $\hat{\mu} \in \mathbb{R}^{|\mathcal{E}|}$ is the estimator state and $\hat{u}$ is the estimator input to be designed. Substituting the above control law to (2) gives us the following autonomous system

$$\dot{p} = v \tag{24}$$

$$\dot{v} = -v - \overline{B}D_z e - \overline{S}_1 D_z(\mu - \hat{\mu}) \tag{25}$$

$$\dot{z} = \overline{B}^T \dot{p} = \overline{B}^T v \tag{26}$$

$$\dot{e} = 2D_z^T \dot{z} = 2D_z^T \overline{B}^T v \tag{27}$$

$$\dot{\hat{\mu}} = \hat{u}. \tag{28}$$

Note that the estimating agents are encoded in S_1, in other words, for the edge $\mathcal{E}_k$ the estimating agent is $\mathcal{E}_k^{\mathrm{tail}}$.

Theorem 7. For the autonomous system (24)-(28) that forms a rigid formation, consider a given desired formation shape and the following distributed control action for the estimator $\hat{\mu}$

$$\hat{u} = -D_z^T \overline{S}_1^T v, \tag{29}$$

where the estimating agents are chosen arbitrarily. Then the equilibrium points $(p^*, v^*, z^*, e^*, \hat{\mu}^*)$ of (24)-(28) are asymptotically stable. Furthermore, $v^* = \mathbf{0}$ and the steady-state deformation of the shape satisfies $||e^*||^2 \leq 2||\mu - \hat{\mu}(0)||^2 + 2||v(0)||^2 + ||e(0)||^2$.

Proof. First we start proving that (29) is a distributed control law. This is clear since the dynamics of $\hat{\mu}_k$ (the k'th element of $\hat{\mu}$) are given by

$$\dot{\hat{\mu}}_k = z_k^T v_{\mathcal{E}_k^{\mathrm{tail}}}, \tag{30}$$

which implies that the estimating agent $\mathcal{E}_k^{\mathrm{tail}}$ for the edge $\mathcal{E}_k$ is only using the dot product of the associated relative position z_k and its own velocity. Note that using the notation in (18), the above estimator input is given by $\alpha_{k\mathcal{E}_k^{\mathrm{tail}}}$. Consider the following Lyapunov function candidate

$$V = \frac{1}{2}||\xi||^2 + \frac{1}{2}||v||^2 + \frac{1}{4}||e||^2,$$

with $\xi = \mu - \hat{\mu}$, which satisfies

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \xi^T \dot{\xi} + v^T \dot{v} + \frac{1}{2} e^T \dot{e} \\
&= \xi^T D_z^T \overline{S}_1^T v - ||v||^2 - v^T \overline{B} D_z e - v^T \overline{S}_1 D_z \xi \\
&+ e^T D_z^T \overline{B}^T v \\
&= -||v||^2.
\end{aligned} \tag{31}$$

From this equality we can conclude that ξ, v and e are bounded. Moreover, from the definition of e, z is also bounded. Thus all the states of the autonomous system (25)-(28) are bounded, so we can conclude the convergence of $v(t)$ to zero in view of (31). Furthermore, since the right-hand side of (25) is uniformly continuous, $\dot{v}(t)$ converges also to zero by Barbalat's lemma. By invoking the LaSalle's invariance principle, looking at (25) the states e, ξ and z converge asymptotically to the largest invariance set given by

$$\mathcal{T} \stackrel{\Delta}{=} \{e, z, \xi : \overline{S}_1 D_z \xi + \overline{B} D_z e = \mathbf{0}_{m|\mathcal{V}|\times 1}\}, \tag{32}$$

in the compact set

$$\mathcal{Q} \stackrel{\Delta}{=} \{\xi, v, e : ||\xi||^2 + ||v||^2 + \frac{1}{2}||e||^2 \leq \rho\}, \tag{33}$$

with $0 < \rho \leq 2V(0)$. Since $v = 0$ for all points in this invariant set, it follows from (26)-(28) that z, e and $\hat{\mu}$ are constant in this invariant set. In other words, $z(t) \to z^*, e(t) \to e^*$ and $\xi(t) \to \xi^*$ as t goes to infinity, where z^*, e^* and $\hat{\mu}^*$ are fixed points satisfying (32). Note that by comparing (24) and (26) we can also conclude that $p(t) \to p^*$ as t goes to infinity. In general we have that e^* and ξ^* are not zero vectors, therefore $z^* \notin \mathcal{Z}$. It is also clear that $||e^*||^2 \leq 2\rho$, therefore for a sufficiently small ρ, the resultant (distorted) formation will also be minimally rigid.

Remark 8. For a triangle (in 2D case) or tetrahedron (in 3D) shape, it can be shown easily that $e^*, \xi^* = \mathbf{0}_{m|\mathcal{E}|\times 1}$.

In comparison to the distributed estimator as proposed in Garcia de Marina et al. (2015a), the proposed of update law of estimator in Theorem 7 (c.f. Eq. (29)) is gain independent as opposed to the one proposed in Garcia de Marina et al. (2015a). In this case, there is no lower bound of estimator gain as required in the aforementioned result. Furthermore, using the proposed method in Theorem 7, we can choose the estimating agents (as embedded in S_1) arbitrarily while the proposed estimator in Garcia de Marina et al. (2015a) has to be chosen carefully. One of the drawback in the proposed estimator above is that the final shape may no longer be the desired shape.

If we adopt the same approach as in Garcia de Marina et al. (2015a), we can guarantee that the mismatches can be fully compensated such that the final shape is the desired shape. We recall the following update law for each estimator as used in Garcia de Marina et al. (2015a) in order to remove effectively both, the distortion and the steady-state collective motion:

$$\hat{u}_k = \kappa(e_k + \mu_k - \hat{\mu}_k), k \in \{1, \ldots, |\mathcal{E}|\}, \tag{34}$$

where $\kappa \in \mathbb{R}^+$ is an estimator gain that needs to be determined. For the first-order agents, it has been shown in Garcia de Marina et al. (2015a) that there is a lower bound for choosing κ.

Consider now the following change of coordinates $h_k = e_k + \mu_k - \hat{\mu}_k$ and let $h \in \mathbb{R}^{|\mathcal{E}|}$ be the stacked vector of h_k's for all $k \in \{1, \ldots, |\mathcal{E}|\}$. By defining $S_2 \stackrel{\Delta}{=} B - S_1$ it can be checked that the following autonomous system derived from (25)-(28)

$$\dot{v} = -v - \overline{S}_2 D_z e - \overline{S}_1 D_z h \tag{35}$$

$$\dot{e} = 2 D_z \overline{B}^T v \tag{36}$$

$$\dot{h} = 2 D_z \overline{B}^T v - \kappa h \tag{37}$$

$$\dot{z} = \overline{B}^T v, \tag{38}$$

has an equilibrium at e^*, v^* and h^* equal to zero with $z^* \in \mathcal{Z}$. The linearization of the autonomous system (35)-(38) about such an equilibrium point leads to

$$\begin{bmatrix} \dot{v} \\ \dot{e} \\ \dot{h} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\overline{I}_{|\mathcal{V}|} & -\overline{S}_2 D_{z^*} & -\overline{S}_1 D_{z^*} & \mathbf{0} \\ 2D_{z^*}\overline{B}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2D_{z^*}\overline{B}^T & \mathbf{0} & -\kappa I_{|\mathcal{E}|} & \mathbf{0} \\ \overline{B}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} v \\ e \\ h \\ z \end{bmatrix}. \tag{39}$$

From the Jacobian in (39) we know that the stability of the system only depends on v, e and h and not on z. We consider the following assumption as in Garcia de Marina et al. (2015a).

Assumption 9. The matrix $\begin{bmatrix} -\overline{I}_{|\mathcal{V}|} & -\overline{S}_2 D_{z^*} \\ 2D_{z^*}\overline{B}^T & \mathbf{0} \end{bmatrix}$ is Hurwitz.

Theorem 10. If Assumption 9 holds then there exists a positive constant κ^* such that the equilibrium of $h = 0, v = 0$ and $e = 0$ (with $z^* \in \mathcal{Z}$) of the autonomous system (25)-(28) with the estimator input $\hat{u}_k$ be given by (34) is locally exponentially stable for all $\kappa > \kappa^* > 0$.

Proof. Taking the Jacobian for v, e and h in (39) along with Assumption 9 as starting point, the main argument of the proof is identical to the one provided in the main result of Garcia de Marina et al. (2015a) and it is omitted here for the sake of brevity.

Remark 11. Since $v(t)$ converges exponentially to zero, it follows immediately that $p(t)$ converges exponentially to a fixed point p^*.

The Assumption 9 is also related to the stability of formation control systems whose graph $\mathbb{G}$ defining the sensing topology is directed. In fact, it is straightforward to check that the matrix in the Assumption 9 is the Jacobian matrix for v and e in a distance-based formation control system (without mismatches) with only directed edges in $\mathbb{G}$. This relation shows how to choose the estimating agents in order to fulfill Assumption 9.

5. SIMULATIONS

In this section we validate the results in Theorem 7 with numerical simulations for a team of four agents where the prescribed shape is a square with side-length of 20 units.

In order to validate Theorem 7 consider the following incidence matrix

$$B = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}, \tag{40}$$

where we set the edge $\mathcal{E}_2$ to be the diagonal of the desired square shape, i.e. $d_2 = 28.28$ units. Note that the estimating agents are chosen according to S_1, which is derived from (40). Let the following randomly generated vector be the mismatches μ_k for each edge $\mathcal{E}_k$

$$\mu = [0.243 \; 0.328 \; 0.397 \; 0.109 \; 0.448]^T. \tag{41}$$

We spread randomly the four agents within an area of 15×15 units and with random initial velocities but with speeds smaller than 2 units per second. We apply the control law as in (25) with the estimator dynamics (29). The results are shown in Figure 3.

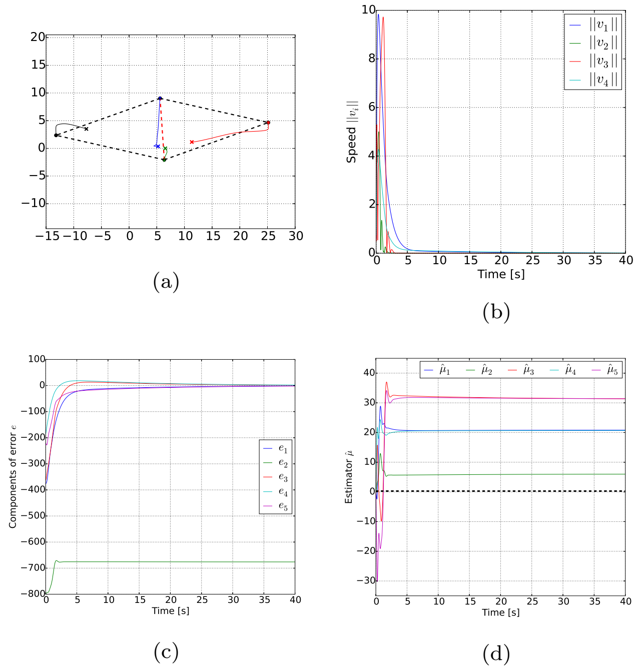

Fig. 3. Numerical simulation of a team of four agents with mismatches in their prescribed distances. We employ the estimator (29). The effectiveness of the estimator is shown in (b) and (c), where all the agents' speed and error signals go to zero excepting for e_2. The steady state of the norm $||e_2^*|| = 676$ squared units is upper bounder practically by the norm of the initial total error $||e(0)|| = 996$ squared units. The plot (d) shows how the estimator state $\hat{\mu}$ does not converge in general to the value of μ (shown in black dashed lines).

We validate Theorem 10 for the same regular squared shape under the sensing topology (40) and mismatches (41). First of all, it can be checked that Assumption 9 is satisfied. We consider the gain $\kappa = 10$ for (34) and as before we spread randomly the four agents within an area of 15×15 units and with random initial velocities but with speeds smaller than 2 units per second. Although the results shown in Figure 4 have better performance than the ones in Figure 3, we want to recall that for applying Theorem 10 we need to in addition satisfy Assumption 9, which in general is possible since it is closely related to the formation control problem with directed graphs, and we need to apply a sufficiently high gain κ.

6. CONCLUSIONS

In this paper we have studied the robustness issue in the application of gradient-based formation control law for second-order agent dynamics under the influence of distance mismatches. It is shown that the closed-loop systems exhibit the same undesired behaviors as in the first-order case, i.e., a stationary distorted shape and undesired collective group motion. Finally, we have proposed two different solutions for compensating the detrimental effects of distance mismatches or constant disturbances in the proximity sensors by employing distributed estimators. We are currently extending the findings in motion control presented in Garcia de Marina et al. (2015b) to second order agents.

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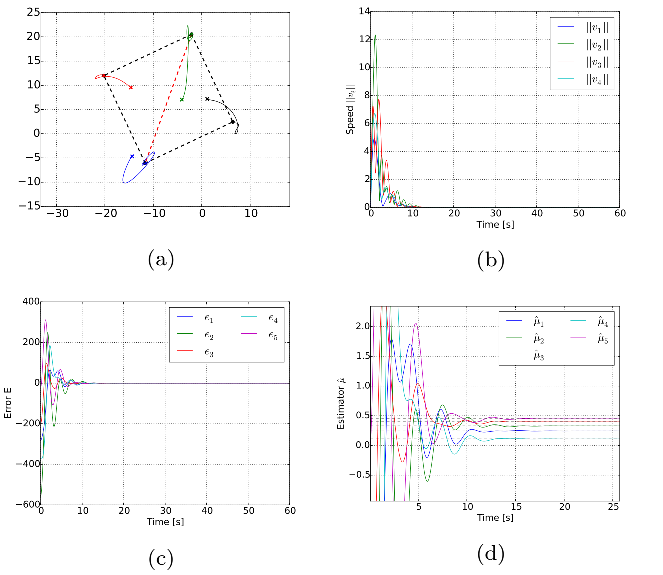

Fig. 4. Numerical simulation of a team of four agents with mismatches in their prescribed distances. We employ the estimator (34) and (40) by checking that we satisfy Assumption 9. We also choose $\kappa = 10$. As it is shown in (a) and (c) the formation converges to the desired regular square and eventually stops as shown in (b). The plot (d) shows how the components of the estimator μ_k converges to $\tilde{\mu}_k$ (shown in black dashed lines).

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